Abstract—The paper presents a brief survey of the methods for solving systems of linear equations (linear solvers), which could be useful in application to WCDMA network optimization. The methods are scored with respect to their computational cost when base stations are equipped with traditional antennas as well as with smart antennas. Preconditioning in the Conjugate Gradient Square (CGS) method is discussed. We also analyze the technique, which without any loss of accuracy, may reduce dimensions of the power control problem in a multi-service network but is limited only to traditional antennas; the smart antenna case must be solved with conventional solver. Theoretical considerations are confirmed experimentally.

Keywords: linear solvers, network optimization, UMTS, smart antenna

I. INTRODUCTION

Considerations presented in this paper are restricted to the WCDMA network optimization at the stage of layout design where optimization of the base stations location [1], number of sectors and antenna types, their azimuth and tilt angles [2] as well as pilot channel power should be done. This kind of optimization is typically based on static system-level simulations. In this approach, the variables of the cost function are usually expressed in terms of transmitted powers that depend on the parameters to be optimized. Excluding very simplified models, transmitted powers usually cannot be presented as analytical functions of the desired parameters. This implies the use of numerical methods for computations of transmit powers. Computing these powers is the most computationally intensive task in the overall optimization problem [5]; thus, finding proper (fast) methods seems to be crucial and required.

II. NETWORK MODEL

In general, transmitted powers are governed by the law of power balance in a transmission link, and they can be numerically computed from equations describing target SIRs (Signal-to-Interference Ratio) at the receiver.

For traditional antennas in the uplink we have for \( \forall k: k \in [1, K, K] \):

\[
\frac{l_k^{(h,b)} p_k^{(b)}}{\sum_{j=1}^{n_k} l_j^{(h,b)} p_j^{(b)} + \left( N_0 B / G \right)} = \gamma_k
\]

where \( \gamma_k = \frac{R_b^{(k)} E^{(k)}}{B} \) is a target SIR of the \( k \)-th mobile station (MS), \( p_k^{(b)} \) is a power of the \( k \)-th MS assigned to the \( h \)-th base station (BS), \( l_k^{(h,b)} \) is an attenuation path between the \( k \)-th MS assigned to the \( h \)-th BS and its BS, \( E^{(k)} \) is the required ratio of bit energy to noise and interference power spectral density, \( B \) is a channel bandwidth, \( N_0 \) is a spectral density of thermal noise power, \( G \) is a directional gain for a pair of BS-MS antennas that are assumed to be omni-directional, \( n_j \) is a number of users in the \( j \)-th cell and \( m \) is a number of cells. Let \( K \) be a total number of users, i.e. \( K = \sum_{j=1}^{m} n_j \).

The equations (1) can be rewritten in the form of the following system of linear equations:

\[
A^{(UL)} p^{(UL)} = b^{(UL)}
\]

where \( p^{(UL)} = [p_1^{(i)}, K, p_1^{(m)}, K, p_m^{(m)}, K] \in \mathbb{R}^K \) is the vector of unknown powers transmitted from mobiles to cells, \( b^{(UL)} = [b_1^{(i)}, K, b_1^{(m)}, K, b_m^{(m)}, K] \in \mathbb{R}^K \) is the noise vector, and

\[
A^{(UL)} = \begin{bmatrix} A^{(UL)}_{(1,1)} & L & A^{(UL)}_{(1,m)} \\ \cdot & \cdot & \cdot \\ A^{(UL)}_{(m,1)} & L & A^{(UL)}_{(m,m)} \end{bmatrix} \in \mathbb{R}^{K \times K}
\]

is a block matrix, where

Application of Linear Solvers to UMTS Network Optimization without and with Smart Antennas

Rafal Zdunek¹, Maciej J. Nawrocki¹², Mischa Dohler², A. Hamid Aghvami²

¹Institute of Telecommunications and Acoustics, Wroclaw University of Technology
Wroclaw, POLAND
²Centre for Telecommunications Research, King’s College London
London, UK
rafal.zdunek@pwr.wroc.pl
maciej.nawrocki@kcl.ac.uk
mischa.dohler@kcl.ac.uk
for $u \neq w$ and $\forall (u, w) \in \{1, K, m\}$, and

$$A^{(u,w)}_{(DL)} = \begin{bmatrix} \begin{pmatrix} f_{i}^{(u,w)} \end{pmatrix} L & f_{i}^{(u,w)} \\ M & O \\ M & f_{i}^{(u,w)} \end{pmatrix} \in \mathbb{R}^{n_u \times n_w}$$

(4)

Remark 1: Assuming that $o_{i}^{(h)} = 1$ for $\forall h \in \{1, K, m\}$ and $\forall i \in \{1, K, n_{h}\}$, we get: $A^{(UL)} = [A^{(DL)}]^T$.

In the smart antenna (SA) technique, BS antennas dynamically adjust their radiation patterns to the served users. The details on this are given e.g. in [6]. We assume simple a SA model where each BS antenna has a 10° beam width and receives the desired signal from a MS with gain $G_o$. The interfering signals within 10° (horizontal) of the main beam of the desired signal are also received with gain $G_o$, and outside of the segment with gain $G_o$ is a side-lobe level. Thus for the SA, we have:

$$G_{DL}^{(b_h)} = \mathcal{F}_{h}_{k}$$

(11)

where $S_{g_{i}}^{(h,j)}$ is a set of indices of the $j$-th BS’s users that interfere the $h$-th BS with gain $G_o$ and $S_{g_{i}}^{(h,k)}$ is a set of indices of the $j$-th BS’s users that interfere the $h$-th BS with gain $G_o$. We have $n_{i} = S_{g_{i}}^{(h,j)} + S_{g_{i}}^{(h,k)}$, where $\mathcal{S}$ denotes a number of entries in set $S$.

Following the similar transformation as for the traditional antennas, the system of linear equations can be derived:

$$A^{(DL)}_{(SA)} p_{(DL)}^{(SA)} = b^{(DL)}_{(SA)}$$

(12)

where $p_{(SA)}^{(DL)} \in \mathbb{R}^{K}$ is the vector of unknown powers transmitted from BSs to mobiles, and $b_{(DL)}^{(SA)} \in \mathbb{R}^{K}$ is the noise vector. Then

Then

$$A^{(DL)}_{(SA)} = \begin{bmatrix} A^{(1,1)}_{(DL)} & \cdots & A^{(1,m)}_{(DL)} \\ \vdots & \ddots & \vdots \\ A^{(m,1)}_{(DL)} & \cdots & A^{(m,m)}_{(DL)} \end{bmatrix} \in \mathbb{R}^{K \times K}$$

(8)

where

$$A^{(u,w)}_{(DL)} = \begin{bmatrix} f_{i}^{(u,w)} \end{pmatrix} L & f_{i}^{(u,w)} \\ M & O \\ M & f_{i}^{(u,w)} \end{pmatrix} \in \mathbb{R}^{n_u \times n_w}$$

(9)

for $u \neq w$ and $\forall (u, w) \in \{1, K, m\}$, and

$$A^{(u,w)}_{(DL)} = \begin{bmatrix} \begin{pmatrix} f_{i}^{(u,w)} \end{pmatrix} L & f_{i}^{(u,w)} \\ M & O \\ M & f_{i}^{(u,w)} \end{pmatrix} \in \mathbb{R}^{n_u \times n_w}$$

(10)

for $\forall u \in \{1, K, m\}$.

### III. METHODS

The systems of linear equations (2), (6) and (12) are square, unsymmetrical and consistent. Our preliminary tests prove that they are also non-singular and quite well-conditioned. Many methods [3], [7] for solving such systems of equations could be applied here but unfortunately the area of application is severely limited if we take into account the computational cost that should be kept as low as possible.
For both directions of signal transmission, our aim is to find the solution to the following system of linear equations at lowest computational cost:

$$A p = b$$  \hspace{1cm} (14)

where $A \in \mathbb{R}^{K \times K}$, $p \in \mathbb{R}^K$ and $b \in \mathbb{R}^K$.

Let us consider an application of the Gauss elimination. The solution can be obtained with $K^3/3 + K^2 - K/3$ multiplications/divisions (m/d) and $K^3/3 + K^2 - 5 \cdot K/6$ additions/subtractions (a/s), without considering the cost of pivoting. In case of very large $K$ (the order of thousands), which takes place in the WCDMA network optimization, the computational cost is prohibitive.

A. Iterative solvers

Another possibility is to apply iterative methods that yield the approximations to the solution. If we do not care much about the accuracy of the final result, the overall computational cost may be kept at reasonable levels. In our considerations, the Richardson, Jacobi, Gauss-Seidel, SOR and Preconditioned CGS methods are compared. The first four ones belong to the class of stationary iterative methods [3], i.e.

$$S p^{k+1} = T p^k + b$$  \hspace{1cm} (15)

where $A = S - T$.

**Theorem 1:** The iterations of (15) are convergent if and only if every eigenvalue $\lambda$ of $S^{-1}T$ satisfies $|\lambda| < 1$. The convergence rate depends on the spectral radius $\rho(S^{-1}T) = \max_i |\lambda_i|$.

The proof of the theorem can be found, e.g., in [3]. Matrix $S$ can be regarded as a left preconditioner, and its selection determines the method as well as its convergence rate. We consider the following cases:

- **Richardson:** $S = I_K \in \mathbb{R}^{K \times K}$ (identity matrix)
- **Jacobi:** $S = \text{diag}(a_{ii})$ (diagonal part of $A$)
- **Gauss-Seidel:** $S = L$ (lower triangular of $A$)
- **SOR:** combination of the last two cases.

The initial guess in all the iterative algorithms discussed here is $p^0 = 0$.

The standard Richardson method is known in the literature to be very slow-convergent, however, with the appropriate preconditioning this method can be competitive.

Our preliminary tests showed that the eigenvalues of $A$ for a typical WCDMA network configuration are very small; hence, it is obvious that the right scaling must be applied to matrix $A$. This can be achieved by a right-side preconditioning, i.e. $(A M^{-1}) (M p) = b$. Let $A M^{-1} = \tilde{A}$ and $M p = \tilde{p}$, thus we have $\tilde{A} \tilde{p} = b$. Matrix $M$ should be defined in this way in order to eigenvalues of $A^{-1}$ would be close to one, and the system $M p = \tilde{p}$ should be very easy to solve. Taking into account this, we decided to make a column scaling, i.e. we took

$$M^{-1} = \text{diag}\left(\frac{1}{\|a_i\|_2}, K\cdot \frac{1}{\|a_i\|_2}\right),$$  \hspace{1cm} (16)

where $a_i$ ($1 \leq i \leq K$) is the $i$-th column of $A$.

Remark 3: The arithmetic operations related to the preconditioning (16) require $(K^2)_{pp} + (K^2)_{pp} + (K)_{dd}$. The implementation of the preconditioned Richardson method is as followed:

$$\begin{align*}
&\text{For } k = 1, 2, \ldots, \text{ do} \\
&\quad \tilde{p}^{k+1} = \tilde{p}^k + \gamma (b - \tilde{A} \tilde{p}^k) \\
&\text{enddo}
\end{align*}$$

where $p = M^{-1} \tilde{p}$, and

$$0 < \gamma < \frac{2}{\lambda_{\max}(A^T A)}$$  \hspace{1cm} (18)

Remark 4: The computational cost of one iteration in the Richardson method is $K_{uu} + K_{pp} + 2K_{dd}$. Preconditioned Richardson method is $K_{uu} + K_{pp} + 2K_{dd} + K_{pp}$, for a typical WCDMA network configuration are very large (an order of thousands), which takes place in the WCDMA network optimization, the computational cost is prohibitive.

Remark 5: One iterative step of the Jacobi’s method implemented by (19) needs $(K^2)_{pp} + (K^2)_{pp} + 2K_{dd}$. The subscript $d$ denotes division.

The implementation of the Gauss-Seidel method is as followed:

$$\begin{align*}
&\text{For } k = 1, 2, \ldots, \text{ do} \\
&\quad p^{k+1} = L_d^{-1} \left( b - U_p p^k \right) \\
&\text{enddo}
\end{align*}$$

where $A$ is decomposed as $A = L + D + U$. Matrices $L$, $D$, $U$ are lower triangular, diagonal and upper triangular of $A$, respectively. In (20), the sign “\text{−}” stands for the Gaussian elimination, and $L_{dd} = L + D$. Note that $L_d$ has each entry above the diagonal equal to zero, which substantially reduce the computational cost.

Remark 6: In one iterative step, the implementation (20) of the Gauss-Seidel method needs $(K^2)_{pp} + (K^2)_{pp} + 2K_{dd}$ operations. The Successive Over-Relaxation (SOR) method was implemented by this algorithm:

$$\begin{align*}
&L_s = D + \omega L, \quad U_s = D - \omega U, \\
&\text{For } k = 1, 2, \ldots, \text{ do} \\
&\quad p^{k+1} = L_s^{-1} (b - \omega b + U_s p^k) \\
&\text{enddo}
\end{align*}$$

where $\omega \in (0,2)$ is a relaxation parameter.

Remark 7: The computational cost of one iterative step of the SOR implemented as in (21) is $(K^2)_{pp} + (K^2)_{pp} + 2K_{dd}$. One must add to this the cost of creating matrices $L_{ss}$, $U_s$, which constitutes $(K^2)_{pp} + (2K_{dd})$ operations.

Beside the above-mentioned iterative methods, there are many others that are robust especially in the context of solving very large PDEs [7]. They may be also attractive for our application due to their low computational complexity. We selected the CGS (Conjugate Gradient Square) that was proposed by Sonneveld [8]. The one-iteration computational cost of the CGS is the lowest. Moreover, our choice is also motivated by the results of our preliminary tests which showed that the so-called serious breakdown (which may occur in this kind of methods) did not happen here at all.

We used the following algorithm of the CGS:
\[ r_0 = \mathbf{0}, \quad \mathbf{e} = [1, K.]^T \in \mathbb{R}^K, \]

\[ r_{k+1} = \mathbf{e}, \quad z_0 = 0, \quad q_0 = r_0, \quad p_0 = 0 \]

For \( k = 1, 2, \ldots, \) do

\[ \beta_k = \frac{r_k^T r_{k-1}}{r_0^T r_{k-2}}, \quad u_k = \beta_k z_{k-1} + r_{k-1}, \]

\[ q_k = u_k - \beta_k (z_{k-1} - u_k), \quad \alpha_k = r_k^T r_{k-1} - r_0^T r_{k-2}, \]

\[ z_k = u_k - \alpha_k \mathbf{A} q_k, \quad p_k = p_{k-1} + \alpha_k (u_k + z_k), \]

\[ r_k = r_{k-1} - \alpha_k \mathbf{A} (u_k + z_k), \quad (22) \]

Remark 8: The algorithm (22) requires \((2mK + (mK)_m)^{(m)}\) arithmetic operations in one iterative step.

We applied this algorithm to the preconditioned system of linear equations.

B. Dimension reduction

The considerations presented here are based on the assumption that the information on signal levels (powers) of BSs is the most important in the optimization. This implies the reduction in a number of the unknowns. For the traditional antennas the system of \( K \) unknowns can be shrunk to a square system of order \( m \). This is particularly noteworthy with respect to an overall computational cost. This approach has been discussed in [4], [5].

In the uplink, we are interested in the distribution of received powers by BSs. Let us define the variable

\[ n_h = \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha_{ij} p_{ij}^{(h)} + (N_0 B)^{(h)}, \quad (23) \]

which means the sum of all the received powers by the \( h \)-th BS. The nominator \( l_{k}^{(h,k)} p_{k}^{(h)} \) in (1) represents a level of the desired signal received from the \( k \)-th user by the \( h \)-th BS. Adding this expression to the denominator in (1), we have:

\[ \frac{r_{k}^{(h,k)} p_{k}^{(h)}}{\sum_{j=1}^{m} \sum_{i=1}^{n} \alpha_{ij} p_{ij}^{(h)} + (N_0 B)^{(h)}} = \bar{\gamma}_k \quad (24) \]

where \( \bar{\gamma}_k = \frac{\gamma_k}{1 + \gamma_k} \) is the Signal-to-Signal-plus-Interference Ratio (SSIR). After replacing the denominator in (24) with (23), we obtain:

\[ p_{k}^{(h)} = \frac{\bar{\gamma}_k}{l_{k}^{(h,k)} r_k} n_h \quad (25) \]

Inserting (25) to (23) we get the following system of linear equations:

\[ \mathbf{G}_{ij} = \bar{\mathbf{n}}, \quad (26) \]

where \( \mathbf{G}_{ij} = [g_{h,j}] \in \mathbb{R}^{m \times m}, \quad r_i = [r_1, K., r_K] \in \mathbb{R}^m \) and

\[ \bar{\mathbf{n}} = [(N_0 B)^{(h)} K]_{ij}, \quad (N_0 B)^{(h)} \in \mathbb{R}^m. \]

Each entry of \( \mathbf{G} \) can be expressed by

\[ g_{h,j} = \delta_{h,j} - \sum_{i=1}^{n} \frac{L_{(h,i)}}{L_{(h,j)}} \bar{\gamma}_i^{(j)}, \quad (27) \]

where \( \delta_{h,j} \) is the Kronecker delta, and \( \bar{\gamma}_i^{(j)} = \frac{\gamma_i^{(j)}}{1 + \gamma_i^{(j)}} \) for \( i \in [1, K.] \) and \( j \in [1, K.] \). Solution \( \mathbf{p} \) can be readily obtained from vector \( \mathbf{r} \) using (20).

Remark 9: The construction of \( \mathbf{G} \) in (26) requires \((2mK + (mK)_m)^{(m)}\) arithmetic operations. Then \( \mathbf{p} \) is computed from \( \mathbf{r} \) with \((2K)^{(m)}\) operations.

Because of separate antenna characteristic for every MS in smart antenna case, matrix \( \mathbf{A} \) does not have any regularity and above reduction can not be used. The only solution is to select the best solver for full linear system.

IV. SIMULATION RESULTS

Our tests are performed for one randomly selected snapshots of the uplink in the WCDMA network with one antenna per BS (traditional omnidirectional and SA). We assumed 1000 users randomly distributed in 104 cells with a mixture of the spatial uniform and skew-Gaussian distributions. Hence we have \( \mathbf{A} \in \mathbb{R}^{1000 \times 1000} \) and \( m = 104 \).

The layout of BSs and MSs is presented in Fig. 1. Half of the users work with a voice service (\( R_b = 12.2 \) kbps), and the other half with a data service (\( R_b = 64 \) kbps). Site-to-site distance equals 2.67 km.

Fig. 1. The layout of simulated network

For this snapshot and the traditional antennas: \( \max_{j} \| \mathbf{p}_j (\mathbf{A}) \|_2 = 2.1 \times 10^{-7} \) and \( \min_{j} \| \mathbf{p}_j (\mathbf{A}) \|_2 = 8.7 \times 10^{-13} \), and for the SA: \( \max_{j} \| \mathbf{p}_j (\mathbf{A}_{SA}) \|_2 = 2.1 \times 10^{-6} \), and \( \min_{j} \| \mathbf{p}_j (\mathbf{A}_{SA}) \|_2 = 9.2 \times 10^{-12} \). Hence according to the Theorem 1 the convergence of the Richardson, Jacobi, Gauss-Seidel and SOR (\( \omega < 2 \)) methods is definitely guaranteed. For the scaled matrix \( \mathbf{\bar{A}} \), \( \lambda \in [1.032, 0.1813] \) for the case of traditional antennas and \( \lambda \in [0.0536, 0.9431] \) for smart antennas. Since the eigenvalues of the scaled matrix are slightly greater than one, the relaxation in the Richardson must be used. This is achieved by satisfying the condition (18). The CGS does not use the scheme (15).

All the iterative algorithms are run until the stopping criterion \( e^k = \| \mathbf{p}^k - \mathbf{p}^{k-1} \|_2 \geq \varepsilon \) is met, where for arbitrary \( \mathbf{u} \):

\[ \| \mathbf{u} \|_\infty = \max_i |u_i| \], and \( \varepsilon \) is a small number greater than zero. We assume that the solution should be computed with the accuracy up to the fifth significant digit, thus \( \varepsilon = 10^{-6} \).

The plots of \( e^k \) versus iterations are illustrated in Fig. 3 and Fig. 4 for the cases of traditional antennas and SAs, respectively.
The dashed horizontal lines in Fig. 2 and Fig. 3 mark the level $10^{-6}$ at which the iterations are stopped. It follows from Fig. 2 that this level or lower is reached by the Richardson, Jacobi, Gauss-Seidel, SOR and Preconditioned CGS after performing 36, 50, 29, 14 and 7 iterations, respectively. For SAs (see Fig. 3), this level is reached within 15, 4, 3, 5 and 3 iterations for respective methods: the Richardson, Jacobi, Gauss-Seidel, SOR and Preconditioned CGS.

![Fig. 2. The history of error $e^k$ versus iterations for the case of traditional antennas.](image)

The similar analysis for the SAs leads to the following estimations of the costs: $32K^2 + 46K$, $8K^2 + 12K$, $9K^2$, $16K^2 + 6K$ and $14K^2 + 67K$ for the Preconditioned Richardson, Jacobi, Gauss-Seidel, SOR and Preconditioned CGS methods.

This comparison shows that the Gauss-Seidel method is the most promising especially for the SA. For the traditional antennas the CGS takes the first place. The costs presented for the Richardson and SOR (Remark 4 and 7) do not include the cost of determining the optimal value of relaxation parameters ($\gamma$ for the Richardson, and $\omega$ for SOR). This requires computation of at least the highest eigenvalue, which is generally very expensive. In our considerations, we took $\gamma = \frac{3}{2\lambda_{\text{max}}}$ for both cases, and $\omega = 1.3$ for the traditional antennas and $\omega = 1.1$ for the SAs.

Thus the Gauss-Seidel method needs $9 \cdot 10^6$ arithmetic operations to perform 3 iterations.

The operations related to the dimension reduction require $3mK + 2K + m$ operations. Assuming also 3 iterations needed for solving the $m \times m$ system with the Gauss-Seidel method, we can roughly estimate the total computational cost as $4.1 \cdot 10^7$ arithmetic operations. Unfortunately, this case is only useful for the traditional antennas.

V. CONCLUSIONS

In this paper, we presented the application of linear solvers to numerical UMTS network optimization process where traditional and smart antennas are used. A network model was built for both antennas together with detailed analysis of possible linear solvers used for the WCDMA network power control problem. Theoretical considerations were verified by simulations.

Concluding, dimension reduction method [4] is the best (fastest) way to compute transmit powers in WCDMA networks where traditional antennas are used at base stations. For the smart antenna case, this reduction cannot be used because of irregular nature of the gain matrix $A$. Simulations show that the old-fashioned Gauss-Seidel algorithm obtains the best performance in the smart antenna scenario.

ACKNOWLEDGEMENTS

Results presented in this paper were obtained as a part of EU FP6 Marie-Curie Intra European Fellowship project OPTIMISM MEIF-CT-2003-501328 as well as Polish State Committee for Scientific Research grant 3 T11D 001 26.

REFERENCES


