Estimation of Quantile Mixtures via L-moments and Trimmed L-moments

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Abstract

Moments or cumulants have been traditionally used to characterize a probability distribution or an observed data set. Recently, L-moments and trimmed L-moments have been noticed as appealing alternatives to the conventional moments. This paper promotes the use of L-moments proposing new parametric families of distributions that can be estimated by the method of L-moments. The theoretical L-moments are defined by the quantile function i.e. the inverse of cumulative distribution function. An approach for constructing parametric families from quantile functions is presented. Because of the analogy to mixtures of densities, this class of parametric families is called quantile mixtures. The method of L-moments is a natural way to estimate the parameters of quantile mixtures. As an example, two parametric families are introduced: the normal-polynomial quantile mixture and the Cauchy-polynomial quantile mixture. The proposed quantile mixtures are applied to model monthly, weekly and daily returns of some major stock indexes.

Key words: order statistics, quantile function, method of moments, mixture models, distribution families, Cauchy distribution, asset return, stock indexes

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The related R package `Lmoments` is available from CRAN.

1 Introduction

L-moments are linear combinations of order statistics. Compared to the conventional moments, L-moments have lower sample variances and are more robust against outliers. Analogously to the conventional moments, the L-moments of order one to four characterize location, scale, skewness and kurtosis, respectively.

The concept of L-moments originates from various disconnected results on linear combinations of order statistics, e.g. (Sillitto, 1969; David, 1968; Chernoff et al., 1967; Greenwood et al., 1979). Hosking (1990) unified the theory of L-moments and provided guidelines for the practical use of L-moments. Since that several authors have applied L-moments in hydrology, meteorology, quality control and engineering (Dewar and Wallis, 1990; Smithers and Schulze, 2001; Adamowski, 2000; Sankarasubramanian and Srinivasan, 1999; Ben-Zvi and Azmon, 1997; Pilon et al., 1991; Pandey et al., 2001; Elamir and Seheult, 2001; Chen and Tung, 2003). Theoretical advances have been developed as well. Elamir and Seheult (2004) derived the exact variance structure of sample L-moments. Mudholkar and Hutson (1998) introduced a modification of L-moments called LQ-moments. Elamir and Seheult (2003) introduced an extension called trimmed L-moments (TL-moments).

L-moments are defined by the quantile function i.e. the inverse of cumulative distribution function (cdf). Formal definition is given in Section 2. The quantile function $Q(u)$ is an increasing function on the interval $u \in [0, 1]$. If $Q(u)$ is differentiable, the probability density function (pdf) as a function of u may be presented as density-quantile function (Parzen, 1979; Jones, 1992)

$$f(Q(u)) = \frac{1}{Q'(u)}. \tag{1}$$

The benefits of defining a distribution by the quantile function become visible in Section 3 where quantile mixtures are introduced.

The method of L-moments is analogous to the method of moments. Sample L-moments are mapped to the model parameters. Usually, the L-moments of order 1 to r are utilized and the number of the L-moments equals to the number of parameters to be estimated. Hosking (1990) presents the L-moment estimators for some common distributions and demonstrates that in some cases, the method of L-moments may give even better fit than the maximum likelihood estimation. Unfortunately, it is impossible to derive closed form L-estimators for many well-known distributions. For instance, Table 2 in (Hosking, 1990) gives only approximate estimators for log-normal, gamma and generalized extreme value (GEV) distributions. Another example is the generalized lambda distribution (GLD) (Dudewicz and Karian, 2000) for which the L-moment estimators were derived in (Karvanen et al., 2002). Although the GLD family is defined by a quantile function, the closed form estimators can be derived only for the symmetric members of the family.

The difficulties in applying the method of L-moments are the motivation for the present work. The goal of this paper is to propose parametric families of distributions whose parameters can be easily estimated by the method of L-moments. The parametric families contain a wide range of different distributions that have practical importance.

The rest of the paper is organized as follows. In Section 2, the concepts of L-moments and TL-moments are reviewed. In Section 3, the idea of quantile mixtures is introduced. Examples are provided in the subsequent sections: The normal-polynomial quantile mixture is presented in Section 4 and L-moments estimators for the model parameters are derived. The Cauchy-polynomial quantile mixture is presented in Section 5 and TL-moments estimators for the model parameters are derived. In Section 6, the proposed quantile mixtures are applied to model the monthly, the weekly and the daily returns of 10 major stock indexes. Finally, the conclusions are presented in Section 7.

2 L-moments and trimmed L-moments

The first four theoretical L-moments can be defined as

$$\begin{aligned} \lambda_1 = E(Y) &= \int_0^1 Q(u)\mathrm{d}u \\ \lambda_2 = \frac{1}{2}E(Y_{2:2} - Y_{1:2}) &= \int_0^1 Q(u)(2u-1)\mathrm{d}u \\ \lambda_3 = \frac{1}{3}E(Y_{3:3} - 2Y_{2:3} + Y_{1:3}) &= \int_0^1 Q(u)(6u^2-6u+1)\mathrm{d}u \\ \lambda_4 = \frac{1}{4}E(Y_{4:4} - 3Y_{3:4} + 3Y_{2:4} - Y_{1:4}) &= \int_0^1 Q(u)(20u^3-30u^2+12u-1)\mathrm{d}u. \end{aligned} \tag{2}$$

The notation $Y_{i:r}$ refers to conceptual sample (Elamir and Seheult, 2003) where r is the sample size and i is the rank in the ordered sample. L-moments exist if and only if the distribution has a finite mean. Furthermore, a distribution with a finite mean is characterized by its L-moments (Hosking, 1990). Analogously to the conventional moments, λ_1 measures the location, λ_2 measures the scaling, λ_3 measures the skewness and λ_4 measures the kurtosis. Scaling invariant measures are obtained by using L-moment ratios defined as

$$\tau_r = \lambda_r/\lambda_2, \qquad r = 3, 4, \dots \tag{3}$$

Especially, the L-skewness

$$\tau_3 = \lambda_3/\lambda_2 \tag{4}$$

and the L-kurtosis

$$\tau_4 = \lambda_4/\lambda_2 \tag{5}$$

are found useful in several applications because they are more reliable than the moment-based skewness and kurtosis.

Recently, Elamir and Seheult (2003) proposed trimmed L-moments (TL-moments) as generalization of L-moments. The trimming is introduced by increasing the conceptual sample size from r to $r + 2t$, where t is the trimming parameter. For instance, if the trimming parameter $t = 1$, the first four TL-moments are

$$\lambda_1^{(1)} = E(Y_{2:3}) = 6\int_0^1 Q(u)u(1-u)\mathrm{d}u \tag{6}$$

$$\lambda_2^{(1)} = \frac{1}{2}E(Y_{3:4} - Y_{2:4}) = 6\int_0^1 Q(u)u(1-u)(2u-1)\mathrm{d}u$$

$$\lambda_3^{(1)} = \frac{1}{3}E(Y_{4:5} - 2Y_{3:5} + Y_{2:5}) = \frac{20}{3}\int_0^1 Q(u)u(1-u)(5u^2-5u+1)\mathrm{d}u$$

$$\lambda_4^{(1)} = \frac{1}{4}E(Y_{5:6} - 3Y_{4:6} + 3Y_{3:6} - Y_{2:6}) = \frac{15}{2}\int_0^1 Q(u)u(1-u)(14u^3-21u^2+9u-1)\mathrm{d}u.$$

TL-moments are more robust than L-moments (Elamir and Seheult, 2003). They exist even if the distribution does not have a mean. More precisely, if $\mathrm{E}(|X|^{\frac{1}{t+1}})$ is finite then the order statistics of order $t+1 \le i \le n-t$ exist (Sen, 1959; David, 1970) and consequently the TL-moments of trimming t exist. Hence, the TL-moments can be used to characterize heavy tailed distributions, such as the Cauchy distribution.

L-moments and TL-moments can be estimated from a sample as linear combinations of order statistics. Elamir and Seheult (2003) present the following estimator for TL-moments ($t = 0$ for the usual L-moments)

$$l_r^{(t)} = \frac{1}{r}\sum_{i=t+1}^{n-t}\left(\frac{\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r+t-1-k}\binom{n-i}{t+k}}{\binom{n}{r+2t}}\right)X_{i:n}. \tag{7}$$

In the equation (7) it is defined $\binom{n}{k} = 0$ if the condition $0 \le k \le n$ is not true.

3 Quantile mixtures

The idea of modeling the quantile function by polynomials was already proposed by Sillitto (1969). We generalize this idea and propose some specific parametric families of distributions. The recent development in the theory of

L-moments and in computational resources allows us to study the properties of these parametric families explicitly.

We define the quantile mixture as follows

$$Q(u) = \sum_{i=1}^{m} a_i Q_i(u), \tag{8}$$

where $Q_i(u)$ is a quantile function and a_i is a model parameter. The necessary and sufficient condition for the parameters a_i is that $Q(u)$ must be a quantile function. If $Q(u)$ is differentiable, this leads to the condition

$$Q^{'}(u) \geq 0 \text{ when } u \in [0, 1]. \tag{9}$$

Notice the analogy between the quantile mixtures and the mixtures of densities that are defined as

$$f(x) = \sum_{i=1}^{m} a_i f_i(x), \tag{10}$$

where $f_i(x)$ is a pdf, $\sum_{i=1}^{m} a_i = 1$ and $a_i \geq 0$.

The motivation for the quantile mixtures is the connection between the L-moments and the model parameters. From the definition of L-moments it follows

$$\lambda_r = \sum_{i=1}^{m} a_i \lambda_r(Q_i), \tag{11}$$

where λ_r is the L-moment of distribution $Q(u)$ and $\lambda_r(Q_i)$ is the L-moment of distribution $Q_i(u)$. This equation gives an easy way to estimate weights a_i in quantile mixtures: the connection between the L-moments and the model parameters is described by linear equations. This connection can be directly applied to estimate the parameters of the quantile mixture (8).

4 Normal-polynomial quantile mixture

In sections 4 and 5, some specific quantile mixtures are proposed. The model parameters can be estimated as linear combinations of L-moments (or TL-moments). The proposed parametric families are built around the normal distribution or the Cauchy distribution and have practical interest as demonstrated in Section 6.

We define the normal-polynomial quantile mixture as follows

$$Q_{NP3}(u) = bQ_N(u) + a_2u^2 + a_1u + a_0, \tag{12}$$

where Q_N is the quantile function of standard normal distribution and b, a_2, a_1, a_0 are the model parameters. The pdf may be presented as a function of u

$$f(Q(u)) = \frac{1}{bQ'_N(u) + a_2 2u + a_1}. \tag{13}$$

The pdf $f(x)$ as a function of x is obtained as follows

1) Solve u from equation $x = Q_{NP3}(u)$.
2) The value of $f(x)$ is given by $f(Q(u))$ defined in (13).

The L-moments of the normal-polynomial quantile mixture are expressed by

$$\begin{aligned}\lambda_1 &= \frac{a_2}{3} + \frac{a_1}{2} + a_0 \\ \lambda_2 &= b\eta_2 + \frac{a_2}{6} + \frac{a_1}{6} \\ \lambda_3 &= \frac{a_2}{30} \\ \lambda_4 &= b\eta_4,\end{aligned} \tag{14}$$

where

$$\eta_2 = \frac{1}{\sqrt{\pi}} \approx 0.5642 \tag{15}$$

$$\eta_4 = (30\pi^{-1} \arctan(\sqrt{2}) - 9)\eta_2 \approx 0.1226\eta_2 \approx 0.06917 \tag{16}$$

are the second and the fourth L-moments of the standard normal distribution. The solution of (14) gives the estimators

$$\begin{aligned}\hat{b} &= \frac{l_4}{\eta_4} \\ \hat{a}_2 &= 30 l_3 \\ \hat{a}_1 &= 6l_2 - 30l_3 - \frac{6l_4\eta_2}{\eta_4} \\ \hat{a}_0 &= l_1 - 3l_2 + 5l_3 + \frac{3l_4\eta_2}{\eta_4}.\end{aligned} \tag{17}$$

The covariance matrix of the estimates $\hat{b}$, $\hat{a}_2$, $\hat{a}_1$, $\hat{a}_0$ can be estimated applying formulas given by Elamir and Seheult (2004). Let $\mathbf{A}$ be a matrix of the coefficients from (17)

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\eta_4} \\ 0 & 0 & 30 & 0 \\ 0 & 6 & -30 & -\frac{6\eta_2}{\eta_4} \\ 1 & 3 & 5 & \frac{3\eta_2}{\eta_4} \end{pmatrix}. \tag{18}$$

Then the estimated covariance matrix of $(\hat{b}, \hat{a}_2, \hat{a}_1, \hat{a}_0)$ is obtained as

$$\hat{\mathbf{\Sigma}}_a = A\hat{\mathbf{\Sigma}}_l A^{\intercal}, \tag{19}$$

where $\hat{\boldsymbol{\Sigma}}_l$ is the estimated covariance matrix of L-moments l_1, l_2, l_3, l_4 calculated as in (Elamir and Seheult, 2004).

In Figure 1 some example pdfs from the normal-polynomial quantile mixture family are plotted. The distributions are standardized to have $\lambda_1 = 0$ and $\lambda_2 = 1/\sqrt{\pi}$. The standard normal distribution is shown in Subplot d). It can be seen that the family contains a wide variety of distributions with different values of kurtosis and skewness. Figure 2 illustrates the range of possible distributions on the (τ_3, τ_4)-plane. The boundary of the family is calculated numerically from the condition (9). For special cases, the exact limits can be derived as

$$0 \leq \tau_4 \leq \frac{6\eta_4}{\eta_2(6 - \pi\sqrt{2})} \approx 0.4724 \text{ when } \tau_3 = 0 \tag{20}$$

$$-\frac{1}{5} \leq \tau_3 \leq \frac{1}{5} \text{ when } \tau_4 = 0. \tag{21}$$

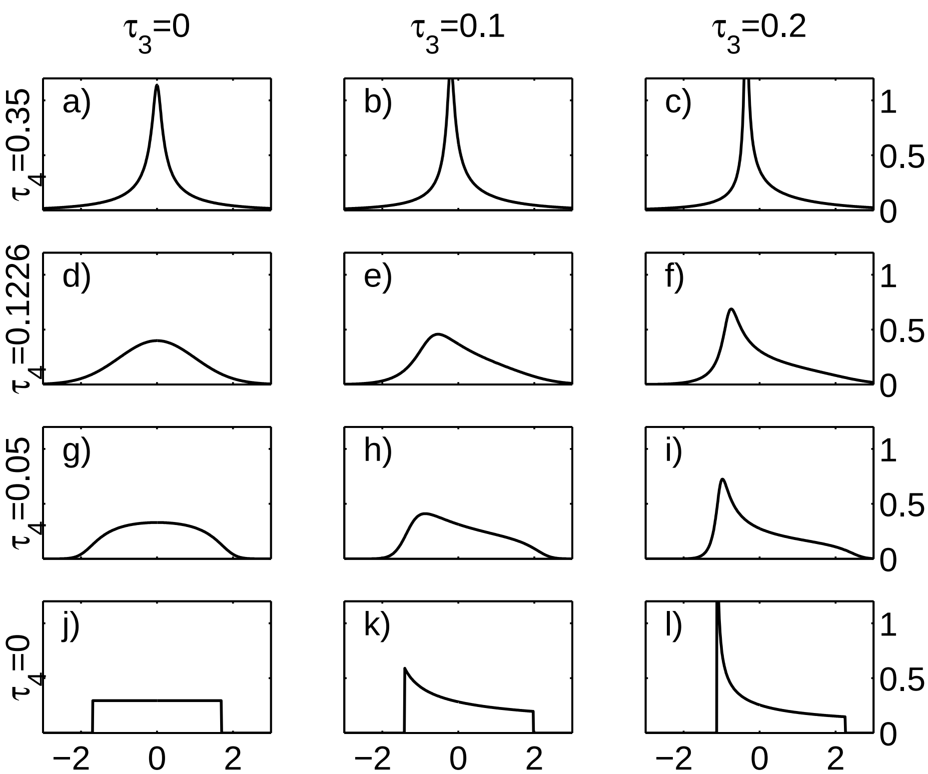


Fig. 1. Some distributions from the normal-polynomial quantile mixture. Each row has the same L-kurtosis and each column the same L-skewness. For instance, in Subplot h) $\tau_4 = 0.05$ and $\tau_3 = 0.1$. The normal(0,1) distribution is shown in Subplot d).

In the equation (12), the standard normal distribution can be replaced with another distribution, which can be, for instance, the Laplacian, the Logistic or the Student's t-distribution. Asymmetric distributions can be also considered.

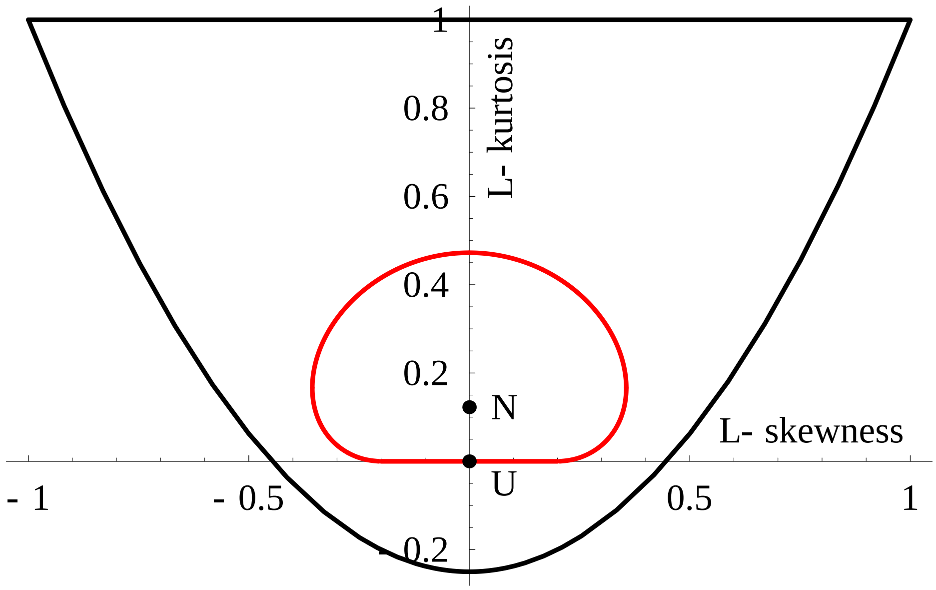


Fig. 2. The L-skewness and the L-kurtosis of the normal-polynomial quantile mixture (12). The outer boundary is the boundary for all distributions and the inner boundary is the boundary for the normal-polynomial quantile mixture. Normal distribution (N) and uniform distribution (U) are members of the parametric family.

In addition, the order of the polynomial in equation (12) can be higher than two. The only requirement is that the number of the L-moments equals to the number of parameters to be estimated. However, the coefficients of the higher order L-moments in the parameter estimators increase quickly as a function of the polynomial order. Thus, the quantile mixtures with high order polynomials may be instable (a small change in the L-moments may cause a big change in the parameters). For instance, if we consider the normal-polynomial quantile mixture of polynomial order 8 ($m = 10$)

$$Q_{NP9}(u) = bQ_N(u) + \sum_{i=0}^{8} a_i u^i, \tag{22}$$

the estimator of a_3 has the following form

$$\hat{a}_3 = 140l_4 - 1260l_5 + 6160l_6 - 21840l_7 + 63000l_8 - 157080l_9 - 128224.2l_{10}. \tag{23}$$

In practice, the problem is not very significant because small values of m already offer sufficient flexibility in the functional forms of quantile mixtures. Value $m = 4$ is often a reasonable choice because there exists a natural interpretation for the four first L-moments (location, scale, skewness, kurtosis).

Normal-polynomial quantile mixture suits well for data generation because the quantile function is explicitly available. Compared to GLD, which also has quantile function explicitly available, the normal-polynomial quantile mixture covers smaller space of L-moments but allows easier control of parameters. The mapping from moments or L-moments to the GLD parameters is not unique and requires use of tables or numerical optimization, whereas the mapping from L-moments to the parameters of the normal-polynomial quantile mixture is unique and linear. There are also differences in the shapes of pdf of the GLD and the normal-polynomial quantile mixture even if the values of four first L-moments are fixed. This is illustrated in Figure 3. The choice between the GLD and the normal-polynomial quantile mixture in a practical data generation problem depends on the problem specific requirements.

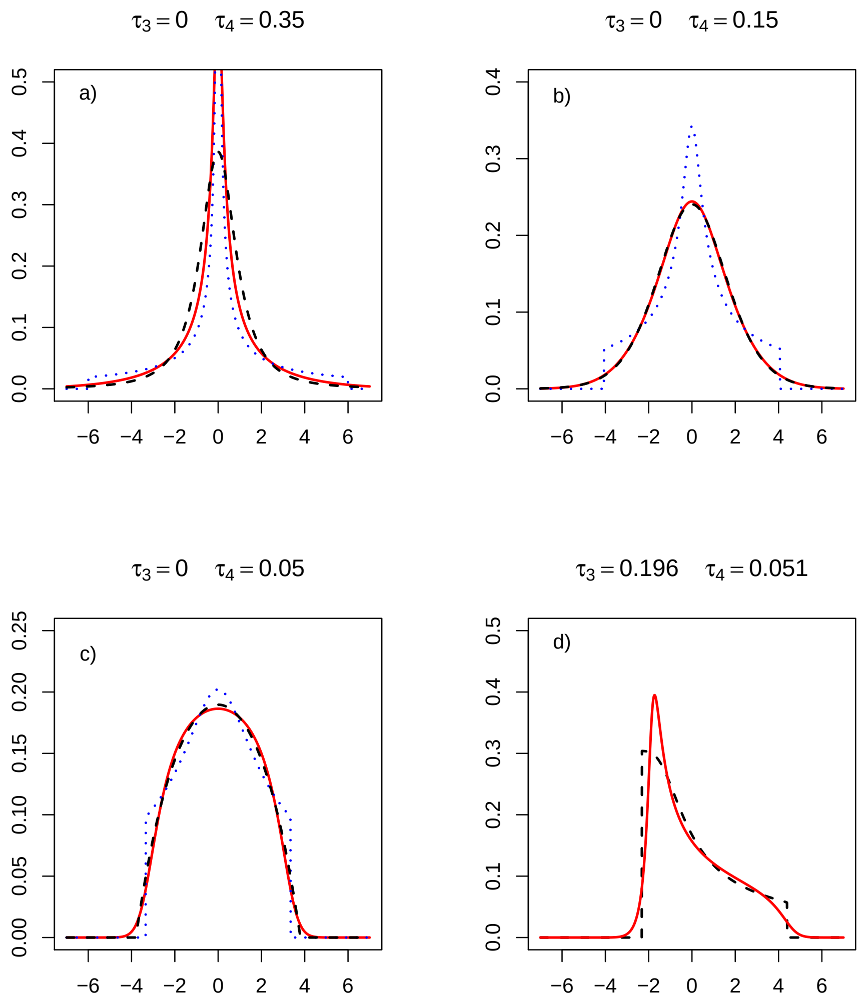


Fig. 3. Illustration of differences in the shapes of pdf of the GLD and the normal-polynomial quantile mixture. The normal-polynomial quantile mixture (solid line) is plotted together with GLDs (dotted and dashed line) sharing the same values of $\lambda_1 = 0$, $\lambda_2 = 1$, τ_3 and τ_4. Despite the fact that the first four L-moments are the same, there are visible differences between the distributions. Only in subplot b), the normal-polynomial quantile mixture and the GLD (dashed line) overlap.

5 Cauchy-polynomial quantile mixture

As the second example, we introduce the Cauchy-polynomial quantile mixture. The model is defined as follows

$$Q_{CP3}(u) = b \tan \pi(u - \frac{1}{2}) + a_2 u^2 + a_1 u + a_0, \tag{24}$$

where $\tan \pi(u - \frac{1}{2})$ is the quantile function of the Cauchy(0,1) distribution and b, a_2, a_1, a_0 are the model parameters. The density function may be presented as a function of u

$$f(Q(u)) = \frac{1}{b\pi \sec^2(\pi(u - \frac{1}{2})) + a_2 2u + a_1}. \tag{25}$$

Because the Cauchy distribution does not have a mean, we must use TL-moments. The TL-moments (t=1) of the Cauchy-polynomial quantile mixture are as follows

$$\begin{aligned} \lambda_1^{(1)} &= \frac{3a_2}{10} + \frac{a_1}{2} + a_0 \\ \lambda_2^{(1)} &= b\nu_2 + \frac{a_2}{10} + \frac{a_1}{10} \\ \lambda_3^{(1)} &= \frac{a_2}{63} \\ \lambda_4^{(1)} &= b\nu_4, \end{aligned} \tag{26}$$

where

$$\begin{aligned} \nu_2 &= \frac{18\zeta(3)}{\pi^3} \approx 0.6978 \\ \nu_4 &= \frac{150\pi^2\zeta(3) - 1575\zeta(5)}{2\pi^5} \approx 0.3428\nu_2 \approx 0.2392 \end{aligned} \tag{27}$$

are the second and the fourth TL-moments (t=1) of the Cauchy(0,1) distribution and ζ is the Riemann zeta function. The solution of (26) gives the estimators

$$\begin{aligned} \hat{b} &= \frac{l_4^{(1)}}{\nu_4} \\ \hat{a}_2 &= 63 l_3^{(1)} \\ \hat{a}_1 &= 10 l_2^{(1)} - 63 l_3^{(1)} - \frac{10 l_4^{(1)} \nu_2}{\nu_4} \\ \hat{a}_0 &= l_1^{(1)} - 5 l_2^{(1)} + \frac{63 l_3^{(1)}}{5} + \frac{5 l_4 \nu_2}{\nu_4}. \end{aligned} \tag{28}$$

In Figure 4, some example pdfs from the Cauchy-polynomial quantile mixture family are plotted. The distributions are standardized to have $\lambda_1^{(1)} = 0$ and $\lambda_2^{(1)} = 0.297$ (the TL-moments of the normal(0,1) distribution). The Cauchy(0,0.426) distribution with TL-moments $\lambda_1^{(1)} = 0$, $\lambda_2^{(1)} = 0.297$, $\tau_3^{(1)} = 0$ and $\tau_4^{(1)} = 0.343$ is shown in Subplot d). The distributions in Figure 4 look relatively similar to the distributions in Figure 1. It is, however, important to notice that in Figure 4 the tails of each distribution are determined by the Cauchy distribution. Consequently, these distributions do not have moments or L-moments.

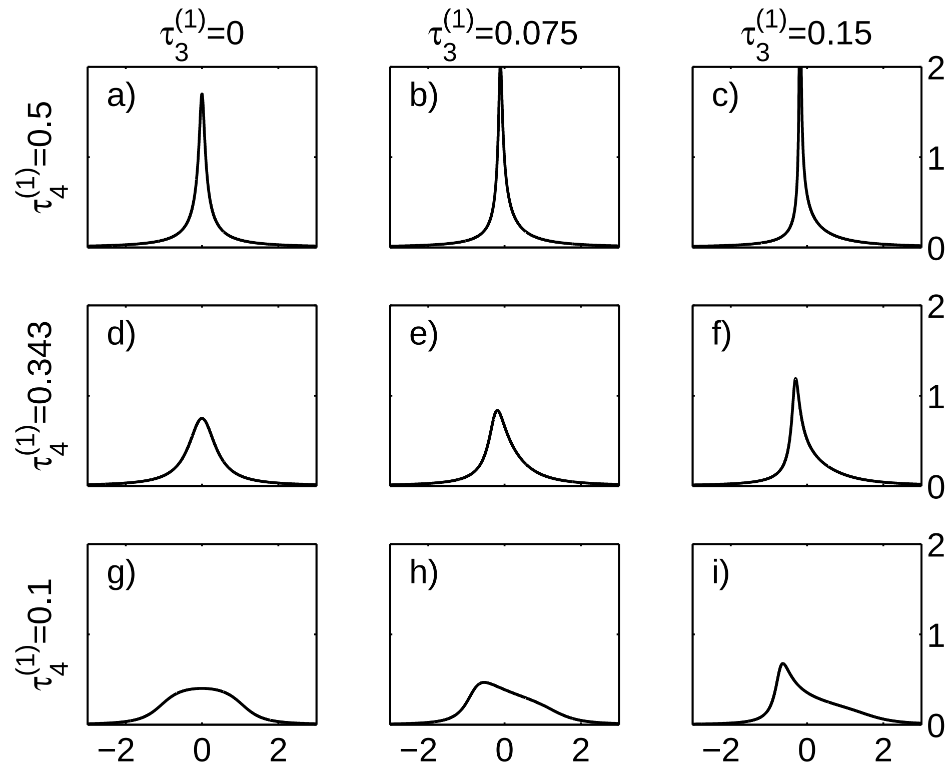


Fig. 4. Some distributions from the Cauchy-polynomial quantile mixture. The distribution in Subplot d) is Cauchy(0,0.426) with TL-moments $\lambda_1^{(1)} = 0$, $\lambda_2^{(1)} = 0.297$, $\tau_3^{(1)} = 0$ and $\tau_4^{(1)} = 0.343$.

The general theoretical boundaries for $\tau_3^{(1)}$ and $\tau_4^{(1)}$ are not presented in the literature. Examining the condition (9), some limits for the Cauchy-polynomial quantile mixture can be derived as:

$$0 \leq \tau_4^{(1)} \leq \frac{10\nu_4}{10\nu_2 - \pi} \approx 0.6235 \text{ when } \tau_3^{(1)} = 0 \tag{29}$$

$$-\frac{10}{63} \leq \tau_3^{(1)} \leq \frac{10}{63} \approx 0.1587 \text{ when } \tau_4^{(1)} = 0. \tag{30}$$

6 Modeling returns of stock indexes

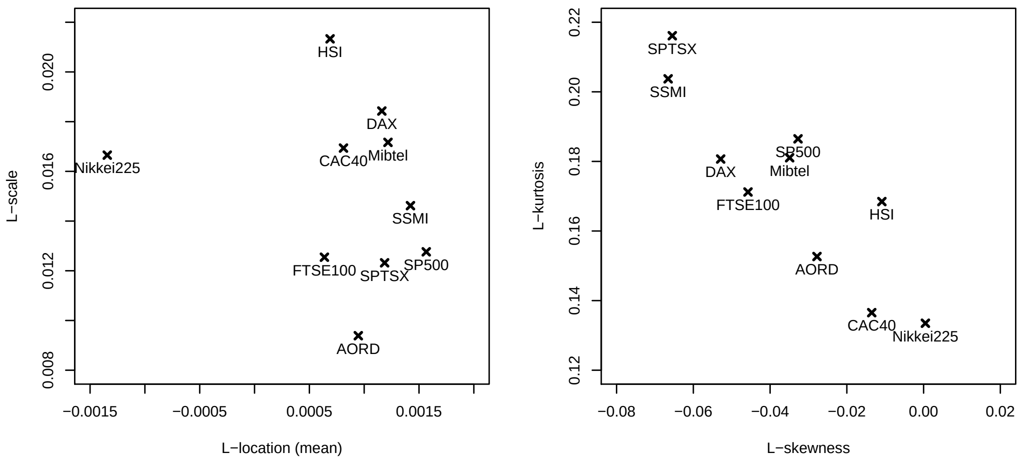
The normal-polynomial quantile mixture (12) and the Cauchy-polynomial quantile mixture (24) are applied to model monthly, weekly and daily returns of 10 major stock indexes: S&P500 (USA), Nikkei225 (Japan), DAX (Germany), AORD (Australia), SPTSX (Canada), CAC40 (France), HSI (Hongkong), Mibtel (Italy), SSMI (Switzerland) and FTSE100 (UK). The data contains monthly, weekly and daily closing values of the indexes from July 1993 to July 2003. Monthly closing values are defined as closing values of each index on the first market day of the month. Weekly closing values are defined as closing values of each index on Mondays. The returns of the indexes are defined as differences of the logarithmic closing values

$$r(t+1) = \log(I(t+1) - \log(I(t))), \tag{31}$$

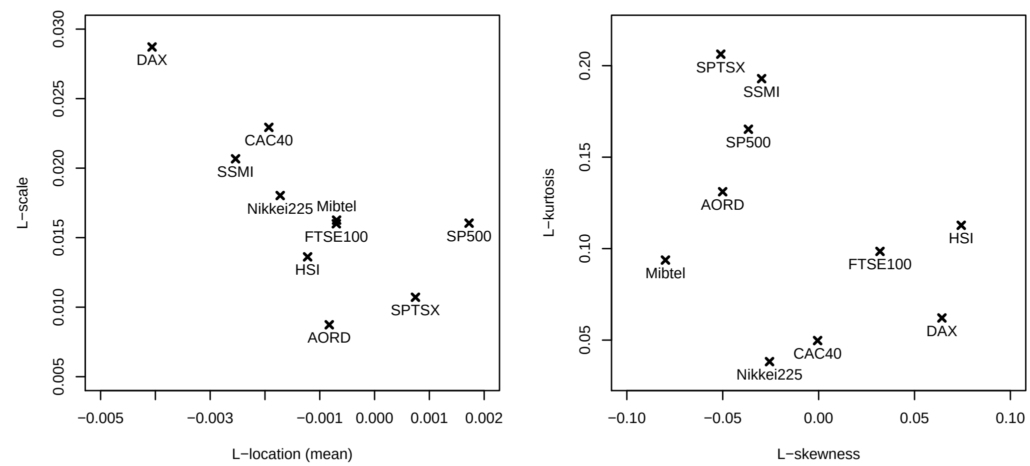
where r is the return and I is the value of the index.

The first example uses two datasets: (a) 519 weekly returns from the 10 year period July 1993 – July 2003 and (b) 52 weekly returns from the 1 year period July 2002 – July 2003. In Figure 5, the returns of stock indexes are characterized by their sample L-moments. Figure 5(a) shows that all indexes except Nikkei225 had positive average return on the studied period of 10 years. The L-scale can be interpreted as a measure of volatility. The L-skewness and the L-kurtosis characterize the shape of the return distribution. The negative values of the L-skewness indicate that big falls are more common than big rises. Figure 5(a) also suggests that there might be a linear dependence between the values of the L-skewness and the L-kurtosis. From Figure 5(b) it can be seen that most indexes had negative average return on the studied 1 year period. In general, the values of L-moments are more scattered than in the data from 519 weeks. There is still some consistency between the periods, for instance, indexes SPTSX and SSMI are characterized by high L-kurtosis both in 519 week data and 52 week data.

The primary goal of this example is to demonstrate the use of the quantile mixtures. The quantile mixtures are compared to the skew t-distribution (Azzalini and Capitanio, 2003). The t-distribution is recognized to have a good fit to stock return data (Blattberg and Gonedes, 1974; Kon, 1984; Töyli, 2002) and skew-t distribution is a generalization of the t-distribution that allows asymmetric distributions. The parameters of the skew-t distribution are estimated using the maximum likelihood approach (employing R package *sn* available from http://azzalini.stat.unipd.it/SN). For the normal-polynomial quantile mixture, the L-moment estimators (17) are used and for the Cauchy-polynomial quantile mixture the TL-moment estimators (28) are used. The results of the model fitting are presented for four stock indexes in Tables 1 and 2. The tables report the estimated parameter values with their standard errors



(a) Weekly returns from 519 weeks



(b) Weekly returns from 52 weeks

Fig. 5. L-moments of the stock index returns.

and the value of the Kolmogorov-Smirnov goodness-of-fit statistics (Conover, 1971). It can be seen from Table 1 that all models provided good fit to the 519 weeks data in terms of the Kolmogorov-Smirnov statistics. The parameter $\hat{b}$ indicates the contribution of the normal component (NP3) or Cauchy component (CP3) in the quantile mixture. As seen from the equations (17) and (28), $\hat{b}$ is determined only by l_4 or $l_4^{(1)}$. The parameters $\hat{a}_2$ and shape of skew-t distribution are related to the skewness of the data. The value of $\hat{a}_2$ is determined only by l_3 or $l_3^{(1)}$. Parameters $\hat{a}_1$ and $\hat{a}_0$ adjust the location and scale but cannot be directly interpreted. Degree of freedom (df) of skew-t distribution indicates how close the distribution is to the skew-normal distribution.

Table 2 summarizes the fitted models estimated from the 52 weeks data. The results reveal a drawback of skew-t distribution. From Figure 5(b) it can seen that several indexes have L-kurtosis smaller than that of normal distribution $\tau_4 = 0.1226$. Two of these indexes, Nikkei225 and DAX are included also in the Table 2. For both of them the estimated degree of freedom (df) is high indicating that in practice the model reduced to the skew-normal distribution. As seen from the value of the Kolmogorov-Smirnov statistics, the skew-normal distribution is not the best possible model for data with small L-kurtosis. The normal-polynomial quantile mixture and the Cauchy-polynomial quantile mixture provided clearly smaller values of the Kolmogorov-Smirnov statistics for Nikkei225 and DAX. If Figure 5 is compared to the Figure 2, it can be seen that all L-skewness–L-kurtosis pairs in Figure 5 are firmly located inside the boundaries of the normal-polynomial quantile mixture.

The second example extends the analysis presented above and studies the quantile mixtures more systematically. Besides weekly stock indexes, we consider also monthly and daily data from the same period of 10 years. Additional datasets are obtained dividing the period of 10 years into non-overlapping subsequences. In total, 1680 datasets are analyzed. The comparison results between skew-t distribution, the normal-polynomial quantile mixture and the Cauchy-polynomial quantile mixture are presented in Table 3. In general, all distributions provided good fit to data. A closer examination reveals that the skew-t distribution provided the best median fit when the number of observations was high (daily 2500, daily 500, daily 250). The Cauchy-polynomial quantile mixture provided the best median fit with weekly data and daily 25 data. The normal-polynomial quantile mixture was good with monthly data. As discussed earlier, the skew-t distribution does not offer good models for samples with small L-kurtosis, which seems to be common when shorter time periods are considered. For longer time periods the skew-t distribution worked well as expected. The differences between the Cauchy-polynomial quantile mixture and the normal-polynomial quantile mixture can be explained by the differences in the tails: the weekly data has heavier tails than the monthly data. For investors the extreme observations have a great importance: they are directly related to the biggest gains and losses in the stock market. Consequently, the L-moments are conceptually better for characterizing stock index returns than the TL-moments: the maximum and the minimum return are not outliers that we should ignore as it is done in the calculation of the TL-moments. Taking this into account, it is rather surprising that the Cauchy-polynomial quantile mixture provided such a good fit to the data. Overall, the results demonstrate the difficulty of the stock index modeling: despite the intensive effort, econometric research has not attained a conclusive view about the best model for stock returns (Töyli, 2002).

Table 1
Models for the weekly returns from 519 weeks. Three models: the normal-polynomial quantile mixture (NP3), the Cauchy-polynomial quantile mixture (CP3) and the skew-t distribution are fitted to four datasets of stock index returns. Estimated parameters and their standard errors (SE) are reported together with the Kolmogorov-Smirnov statistics (K-S).

Index	NP3			CP3		skew-t		
		estim.	SE		estim.		estim.	SE
SP500	K-S	**0.0271**		K-S	0.0284	K-S	0.0286	
	$\hat{b}$	0.0344	2.5e-05	$\hat{b}$	0.00222	locat.	0.00243	0.0050
	$\hat{a}_2$	-0.0125	0.011	$\hat{a}_2$	-0.00842	scale	0.0187	0.00099
	$\hat{a}_1$	-0.0274	0.012	$\hat{a}_1$	0.0552	shape	-0.0229	0.30
	$\hat{a}_0$	0.0194	0.0032	$\hat{a}_0$	-0.0231	df	5.31	1.2
Nikkei	K-S	0.0192		K-S	**0.0163**	K-S	0.0250	
	$\hat{b}$	0.0321	2.7e-05	$\hat{b}$	0.00188	locat.	-0.00143	0.014
	$\hat{a}_2$	0.000236	0.012	$\hat{a}_2$	-0.00215	scale	0.0272	0.0013
	$\hat{a}_1$	-0.00911	0.012	$\hat{a}_1$	0.0756	shape	0.00296	0.60
	$\hat{a}_0$	0.00313	0.0030	$\hat{a}_0$	-0.0385	df	11.9	4.9
DAX	K-S	0.0285		K-S	0.0303	K-S	**0.0210**	
	$\hat{b}$	0.0481	4.0e-05	$\hat{b}$	0.00321	locat.	0.016	0.0057
	$\hat{a}_2$	-0.0292	0.016	$\hat{a}_2$	-0.0249	scale	0.0299	0.0029
	$\hat{a}_1$	-0.0231	0.018	$\hat{a}_1$	0.0931	shape	-0.639	0.25
	$\hat{a}_0$	0.0225	0.0047	$\hat{a}_0$	-0.0369	df	5.62	1.3
AORD	K-S	0.0249		K-S	0.0220	K-S	**0.0207**	
	$\hat{b}$	0.0207	1.6e-05	$\hat{b}$	0.00141	locat.	0.00739	0.0048
	$\hat{a}_2$	-0.00782	0.0068	$\hat{a}_2$	-0.00494	scale	0.0162	0.0020
	$\hat{a}_1$	-0.00599	0.0076	$\hat{a}_1$	0.0429	shape	-0.515	0.37
	$\hat{a}_0$	0.00655	0.0020	$\hat{a}_0$	-0.0187	df	10.2	4.0

7 Conclusion

In this paper, a concept for parametric modeling of a univariate distribution is presented. A quantile mixture is defined as a linear combination of quantile functions. The motivation for quantile mixtures is the fact that the mixture parameters can be estimated as linear combinations of sample L-moments or sample TL-moments. Because L-moments have lower sample variance than

Table 2
Models for the weekly returns from 52 weeks. Three models: the normal-polynomial quantile mixture (NP3), the Cauchy-polynomial quantile mixture (CP3) and the skew-t distribution are fitted to four datasets of stock index returns. Estimated parameters and their standard errors (SE) are reported together with the Kolmogorov-Smirnov statistics (K-S). The applied maximum likelihood procedure did not return standard errors for the skew-t parameters with AORD index.

Index	NP3			CP3		skew-t		
		estim.	SE		estim.		estim.	SE
SP500	K-S	0.0830		K-S	**0.0753**	K-S	0.0951	
	$\hat{b}$	0.0383	8.8e-05	$\hat{b}$	0.00223	locat.	0.00855	0.038
	$\hat{a}_2$	-0.0176	0.037	$\hat{a}_2$	-0.0276	scale	0.0259	0.0088
	$\hat{a}_1$	-0.0159	0.039	$\hat{a}_1$	0.0924	shape	-0.304	1.8
	$\hat{a}_0$	0.0155	0.01	$\hat{a}_0$	-0.0356	df	8.91	10
Nikkei	K-S	**0.0693**		K-S	0.0769	K-S	0.0993	
	$\hat{b}$	0.00997	4.9e-05	$\hat{b}$	0.000385	locat.	-0.0017	0.0049
	$\hat{a}_2$	-0.0139	0.033	$\hat{a}_2$	-0.0209	scale	0.0309	0.0030
	$\hat{a}_1$	0.0883	0.033	$\hat{a}_1$	0.128	shape	-0.000707	0.095
	$\hat{a}_0$	-0.0412	0.0061	$\hat{a}_0$	-0.0588	df	1965	22
DAX	K-S	**0.0467**		K-S	0.0484	K-S	0.0832	
	$\hat{b}$	0.0258	0.00011	$\hat{b}$	0.00148	locat.	-0.00446	0.0069
	$\hat{a}_2$	0.0554	0.050	$\hat{a}_2$	0.0433	scale	0.0494	0.0048
	$\hat{a}_1$	0.0297	0.050	$\hat{a}_1$	0.108	shape	0.010	0.028
	$\hat{a}_0$	-0.0373	0.0084	$\hat{a}_0$	-0.0729	df	2288	30
AORD	K-S	**0.0613**		K-S	0.0795	K-S	0.0724	
	$\hat{b}$	0.0166	4.8e-05	$\hat{b}$	0.000893	locat.	-0.000782	
	$\hat{a}_2$	-0.0131	0.019	$\hat{a}_2$	-0.0238	scale	0.0149	
	$\hat{a}_1$	0.00948	0.018	$\hat{a}_1$	0.0631	shape	-0.000499	
	$\hat{a}_0$	-0.00120	0.0044	$\hat{a}_0$	-0.0248	df	33.6	

the conventional moments, the method of L-moments is expected to result more reliable models than the method of moments. Using TL-moments it is possible to model distributions that do not have moments. We introduced the normal-polynomial quantile mixture and the Cauchy-polynomial quantile mixture and derived the L-moment or TL-moment estimators for them. These parametric families are built around the normal distribution or the Cauchy

Table 3
Comparison between the quantile mixtures and the skew-t distribution in the modeling of monthly, weekly and daily returns of stock indexes. The indexes are listed at the beginning of the Section 6. The data from a period of ten years is divided into subsequences of different length. The column N tells the number of observations in the datasets. For instance, the second row of the table tells that the datasets are monthly returns from periods of 24 months and the number of datasets is 50 (10 stock indexes and 5 periods). Column *median K-S* reports the median Kolmogorov-Smirnov goodness-of-fit statistics of the quantile mixtures and skew-t distributions. Smaller values of K-S statistics indicate better fit to the data. Column *best model* presents the number of cases when each model provided the best fit to the data in the terms of K-S statistics. There were 9 datasets where the maximum likelihood estimation of skew-t distribution did not converge.

Data type	N	median K-S			best model			number of
		NP3	CP3	skew-t	NP3	CP3	skew-t	datasets
Monthly	120	**0.0456**	0.0464	0.0477	4	3	3	10
Monthly	24	0.0972	**0.0935**	0.127	23	21	6	50
Monthly	12	**0.144**	0.147	0.157	45	28	27	100
Weekly	519	0.0276	**0.0231**	0.0235	1	4	5	10
Weekly	104	0.0468	**0.0429**	0.0543	12	32	6	50
Weekly	52	0.0702	**0.0628**	0.0751	27	58	15	100
Weekly	31	0.0949	**0.0866**	0.1060	58	113	29	200
Daily	2500	0.0139	0.0200	**0.0121**	4	0	6	10
Daily	500	0.0254	0.0281	**0.0253**	20	8	22	50
Daily	250	0.0345	0.0343	**0.0325**	30	24	46	100
Daily	25	0.0970	**0.0887**	0.108	321	547	132	1000

distribution and they can model a wide range of symmetric and asymmetric distributions.

We applied the proposed quantile mixtures to stock index data. In total, 1680 datasets were modeled. The results suggest that both the normal-polynomial quantile mixture and the Cauchy-polynomial quantile mixture are good alternatives for modeling stock index data. It was found the quantile mixtures provided better fit to data than the skew-t distribution when the sample size is small. The modeling of the short-term return distribution has also practical importance because in the stock markets the recent data is more important than old data. The results encourage studying the further use of L-moments and quantile mixtures with stock market data.

We argue that if L-moments are used as descriptive statistics of the data, it is natural to use a quantile mixture as a model of the data. The type of quantile mixture should be chosen to reflect the prior knowledge on the data. Introducing new quantile mixtures for specific applications is a potential direction for future work.

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